CHAPTER 16 – VECTOR CACULUS

Reviewing the function types we have learned:

Scalar Functions:

(1)

(2)

Vector Functions:

(3)

(4)

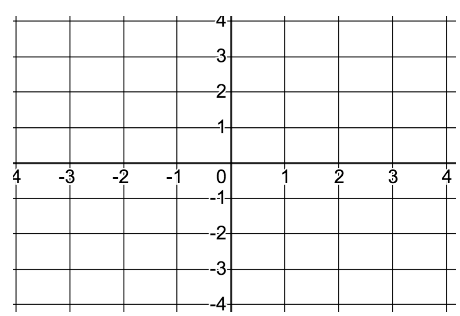
16.1: Vector Fields

> **1 Definition** Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

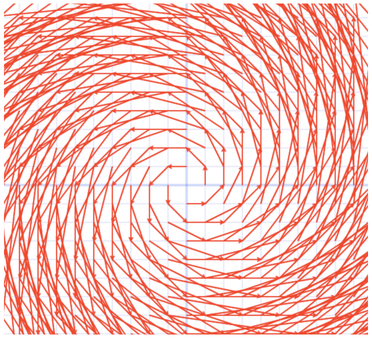
$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = \langle P(x, y), Q(x, y) \rangle$$

Extends to R3: $\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$

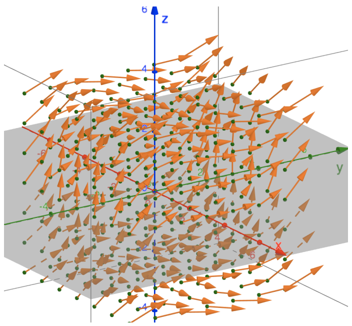
<u>Example</u>: $\vec{F}(x,y) = -y\,\vec{i} + x\,\vec{j} = \langle -y, x \rangle$



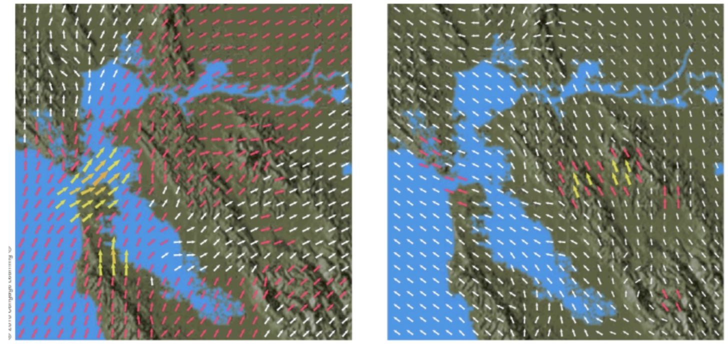
See Vector Field Plotter on 5C page



Velocity Field of Rotating Body



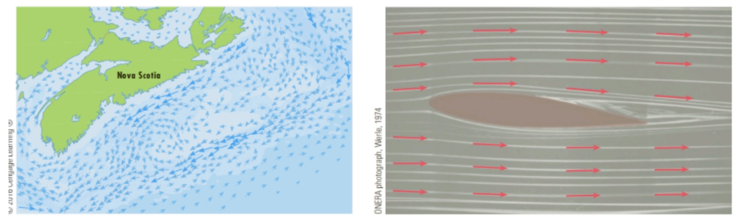
Physical Applications: force fields, gravitational fields, electric fields…



(a) 6:00 PM, March 1, 2010 (b) 6:00 AM, March 1, 2010

FIGURE 1 Velocity vector fields showing San Francisco Bay wind patterns

Other examples of velocity vector fields are illustrated in Figure 2: ocean currents and flow past an airfoil.



(a) Ocean currents off the coast of Nova Scotia (b) Airflow past an inclined airfoil

A Vector Field we have already encountered:

Given $z = f(x,y)$, $\vec{\nabla} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$ is a vector field.

A vector field $\vec{F}(x,y)$ is called ______________________ if it is the gradient of some scalar function $f(x,y)$ called the potential function

Example: If $f(x,y) = x^2y - y^3$, then $\vec{F}(x,y) = \vec{\nabla} f(x,y) =$ ______________________ is conservative and _______________ is the potential function for $\vec{F}(x,y)$

Note: later in the chapter, we will learn how to find the potential function for a given conservative vector field.

Hint on homework matching problems:

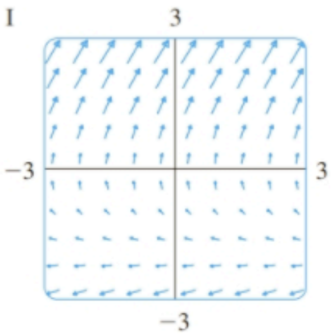
11. $\mathbf{F}(x, y) = \langle x, -y \rangle$

12. $\mathbf{F}(x, y) = \langle y, x - y \rangle$

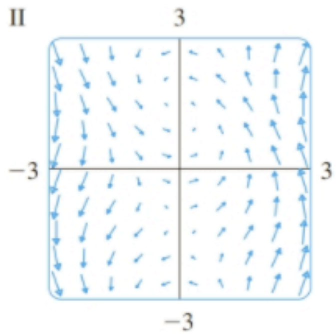
13. $\mathbf{F}(x, y) = \langle y, y + 2 \rangle$

14. $\mathbf{F}(x, y) = \langle \cos(x + y), x \rangle$

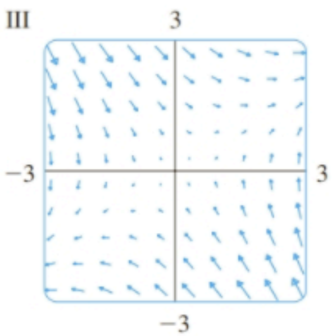
I



II



III



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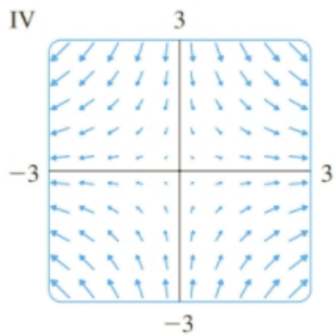


16.2 part ii: Application of Line integral: Work

Force Applied in the Direction of Motion
(Review Section 5.4

Constant Force

Variable Force.

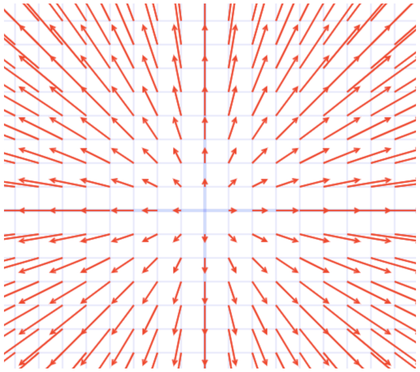
Force Not Applied in the Direction of Motion

Constant Force

Variable Force

Suppose $\vec{F}(x,y)=\vec{P}(x,y)\vec{i}+\vec{Q}(x,y)\vec{j}$ is a continuous vector field (physically, a force field such as gravitational or electric force field) How can we compute the work done by the vector field in moving a particle along a smooth curve given by $\vec{r}(t)=\langle x(t),y(t)\rangle$, $a\le t\le b$

(The derivation for R3 is similar).



So Work = $\int_C \vec{F} \bullet \vec{T}\, ds$ Book notation: Work= $\int_C \vec{F} \bullet d\vec{r}$

Example: Find the work done by the vector field $\vec{F}(x,y) = \langle xy, 2x - y \rangle$ in moving a particle along the curve given by $\vec{r}(t) = \left\langle t, t^2 \right\rangle$; $0 \le t \le 1$

Shortcut:

So by definition: Work = $\int_C \vec{F} \bullet \vec{T}\, ds$ For computation: Work= $\int_a^b \vec{F} \bullet \vec{r}'\, dt$ Book shorthand notation: Work= $\int_C \vec{F} \bullet d\vec{r}$

In General for $\vec{F}(x,y) = \vec{P}(x,y)\vec{i} + \vec{Q}(x,y)\vec{j}$ in R^2 with $\vec{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$

Work = $\int_C \vec{F} \bullet \vec{T}\, ds = \int_a^b \vec{F} \bullet \vec{r}'\, dt =$ ____ "differential form"

Extends to R^3

Example: Find the work done by the force field $\vec{F}(x,y,z) = \left\langle x^2, y^2, z^2 \right\rangle$ on a particle that moves along the line segment from (1,2,-1) to (3,2,0), then from (3,2,0) to (3,2,1).

Alternate notation: Find $\int_C x^2 dx + y^2 dy + z^2 dz$

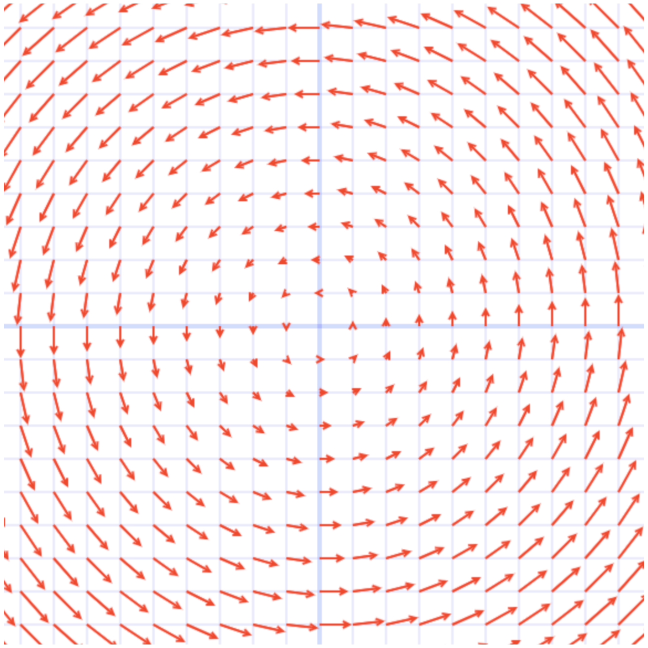
16.3 More methods of computing the WORK line integral – Fundamental Theorem of Line Integrals

Recall from last time: to compute the work done by the vector field in moving a particle along a smooth curve given by $\vec{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$

By definition: Work = $\int_C \vec{F} \bullet \vec{T}\, ds$ For computation: Work= $\int_a^b \vec{F} \bullet \vec{r}'\, dt$ Book shorthand notation: Work= $\int_C \vec{F} \bullet d\vec{r}$

Example:

(a) Find the work done by the vector field $\vec{F}(x,y) = \langle -y, x \rangle$ in moving a particle from (0,0) to (1,1) along the curve $y = x^2$



(b) Changing path, let the particle from (0,0) to (1,1) along the curve $y = \sqrt{x}$

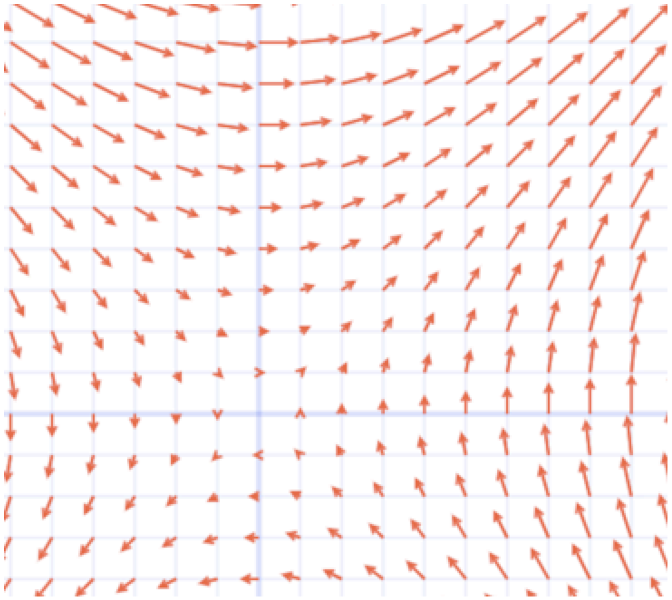
Just for variety, use "creative" parameterization, $\begin{cases} x = 4t \\ y = 2\sqrt{t} \\ 0 \le t \le \frac{1}{4} \end{cases}$

Observations:
Positive Work
Negative Work
Reverse direction changes sign
Does not depend on parameterization
Does depend on path

Example:

(a) Find the work done by the vector field $\vec{F}(x,y) = \langle y,x \rangle$ in moving a particle from (0,0) to (1,1) along the curve $y = x^2$

(b) Changing path, let the particle from (0,0) to (1,1) along the curve $y = \sqrt{x}$



Fundamental Theorem of Line Integrals (book states differently)

Suppose $\vec{F}(x,y)=\langle P(x,y),Q(x,y)\rangle$ where P and Q are continuous in some open region containing a piecewise smooth curve C which starts at (x_a,y_a) and ends at (x_b,y_b), if $\vec{F}(x,y)=\vec{\nabla}f(x,y)$ for some $f(x,y)$ at each point in the region, then

$$\int_C \vec{F}\bullet d\vec{r} = \underline{\hspace{6cm}}$$

That is, if $\vec{F}(x,y)$ is ________________________, we can evaluate the work integral by evaluating the____________________ function at the endpoints and subtracting. (Note the similarity to the Fundamental Theorem of Calculus) Extends to R^3.

Proof: For a smooth curve C given by $\vec{r}(t)=\langle x(t),y(t)\rangle;\quad a\le t\le b$ where $(x_a,y_a)=\vec{r}(a)$ and $(x_b,y_b)=\vec{r}(b)$, and conservative vector field $\vec{F}(x,y)=\vec{\nabla}f(x,y)=\left\langle \frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right\rangle$,

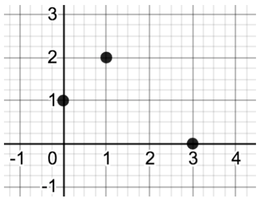
$$\int_C \vec{F}\bullet d\vec{r} = \int_a^b \vec{F}\bullet\vec{r}\,'dt =$$

On previous example, note that $\vec{F}(x,y)=\langle y,x\rangle=\vec{\nabla}f(x,y)$ where $f(x,y)=xy$. So we can compute

$$\int_C \vec{F}\bullet d\vec{r} = f(1,1)-f(0,0)$$

This idea is called ____

Suppose for the same $\vec{F}(x,y)=\langle y,x\rangle$, we wanted to compute the work in moving a particle around the piecewise smooth curve from (1,2) to (0,1) to (3,0) and back to (1,2)



So, if $\vec{F}(x,y)$ is conservative, we have two additional options for how we compute $\int_C \vec{F}\bullet d\vec{r}$

1) ____

2) ____

Note, if the curve is closed, ______________________________

How do we know if $\vec{F}(x,y)$ is conservative and if so, how do we find the potential function ______________________?

First, notice:

If $\vec{F}(x,y)$ is conservative, then $\vec{F}(x,y)=\langle P(x,y),Q(x,y)\rangle$ can be written $\vec{F}(x,y)=\vec{\nabla}f(x,y)=$ ________________

5 Theorem If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

But that is not quite what we need. The converse of this theorem is only true in a special type of region.

Conservative Field Test for R^2

6 Theorem Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

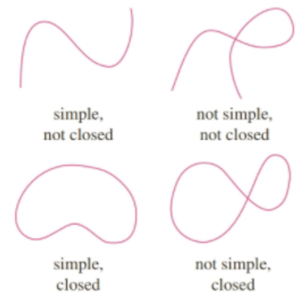


FIGURE 6
Types of curves

From previous examples:

$\vec{F}(x,y) = \langle -y, x \rangle$ $\qquad$ $\vec{F}(x,y) = \langle y, x \rangle$

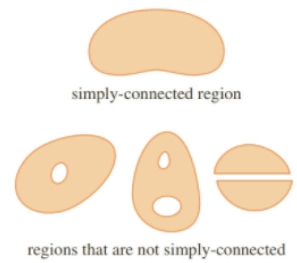


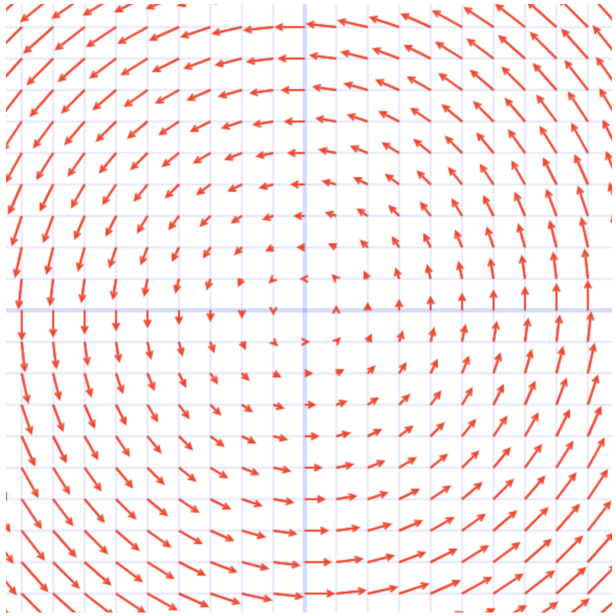
FIGURE 7

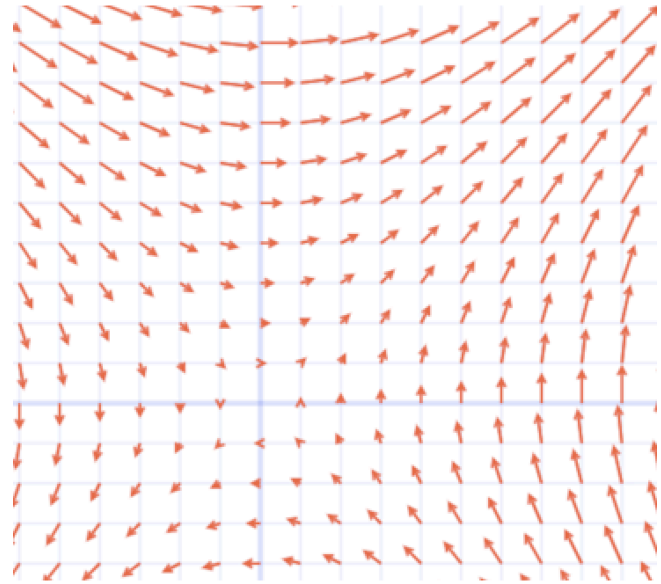
Example: Show that $\vec{F}(x,y) = \langle 3 + 2xy, x^2 - 3y^2 \rangle$ is conservative and find the potential function.

One example, three ways

Find the work done by the vector field $\vec{F}(x,y)=\langle e^y, xe^y\rangle$ in moving a particle along the curve given by $\vec{r}(t)=\langle \cos t, \sin t\rangle; \quad 0\le t\le \pi$

Identifying conservative vector fields from plot.





Conservative Field Test

29. Show that if the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is conservative and P, Q, R have continuous first-order partial derivatives, then

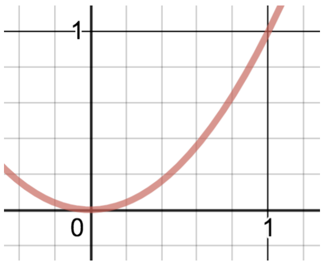
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

Example: Find the work done by the vector field $\vec{F}(x,y,z) = \left\langle y^2z + 2xz^2, 2xyz, xy^2 + 2x^2z \right\rangle$ in moving a particle along the curve given by $\vec{r}(t) = \left\langle \sqrt{t}, t+1, t^2 \right\rangle; \quad 0 \le t \le 1$

16.4 Green's Theorem

Lead in problem:

Compute $\int_C x^2y^2\,dx + xy\,dy$, where C consists of the arc of the parabola $y=x^2$ from (0,0) to (1,1) followed by the line segments from (1,1) to (0, 1) and then (0,1) to (0,0).



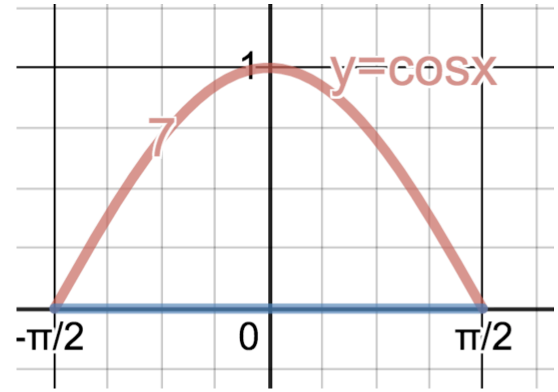
Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Notes:

- See proof in book.
- Green's Theorem can be use reverse to compute double integrals and area.
- Green's Theorem can be extended to more complicated regions.

Example: Find the work done by the vector field $\vec{F}(x,y) = \langle e^{-x} + y^2, e^{-y} + x^2 \rangle$ in moving a particle along the piecewise smooth curve shown.



Two operations on vector fields. Useful in applications of vector calculus including fluid flow, electricity and magnetism. See 5C page for physical description and applications.

Curl

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

1

$$\text{curl}\,\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Divergence

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of F** is the function of three variables defined by

9

$$\text{div}\,\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Notation to help remember these formulas. Define the "del operator" $\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

Gradient: $\vec{\nabla} \quad = \frac{\partial}{\partial x} \quad \vec{i} + \frac{\partial}{\partial y} \quad \vec{j} + \frac{\partial}{\partial z} \quad \vec{k}$

Curl: $curl\,\vec{F} = \vec{\nabla} \times \vec{F}$

Divergence: $div\,\vec{F} = \vec{\nabla} \bullet \vec{F}$

Example: For $\vec{F} = \langle yz,\ xz\ ,\ xy + 2z \rangle$, compute $curl\,\vec{F}$ and $div\,\vec{F}$

Test for Conservative Vector Field in R^3,

Recall from last section:

29. Show that if the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is conservative and P, Q, R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

4 Theorem If **F** is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then **F** is a conservative vector field.

Physical Description of Curl and Div: See CalcPlot 3D, Plot vector field, show curl.

The reason for the name *curl* is that the curl vector is associated with rotations. One connection is explained in Exercise 37. Another occurs when **F** represents the velocity field in fluid flow (see Example 16.1.3). Particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of curl $\mathbf{F}(x, y, z)$, and the length of this curl vector is a measure of how quickly the particles move around the axis (see Figure 1). If curl $\mathbf{F} = \mathbf{0}$ at a point P, then the fluid is free from rotations at P and **F** is called **irrotational** at P. In other words, there is no whirlpool or eddy at P. If curl $\mathbf{F} = \mathbf{0}$, then a

Again, the reason for the name *divergence* can be understood in the context of fluid flow. If $\mathbf{F}(x, y, z)$ is the velocity of a fluid (or gas), then div $\mathbf{F}(x, y, z)$ represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume. In other words, div $\mathbf{F}(x, y, z)$ measures the tendency of the fluid to diverge from the point (x, y, z). If div $\mathbf{F} = 0$, then **F** is said to be **incompressible**.

16.7ii Examples: FLUX Surface Integral in a VECTOR FIELD over a Surface given by a FUNCTION (without parametric surfaces)

Recall, 16.7i, surface integral of a scalar function:

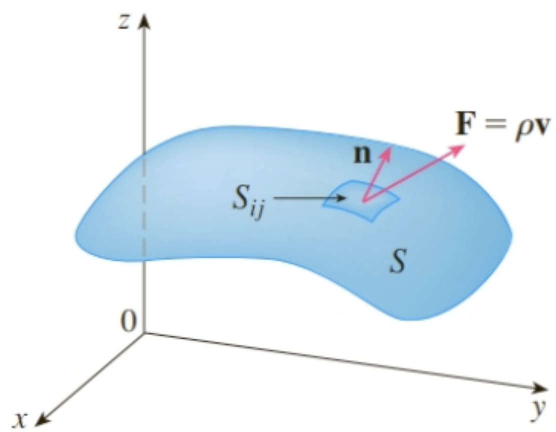
For surface S given by z=g(x,y) over a region D in the xy plane:

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^{\,2}+g_y^{\,2}+1}\;dA$$

Similarly for the other two orientations y=g(x,z), x=g(y,z)

Here we will consider an application of the Surface Integral

Suppose S is a surface in a vector field $\vec{F}(x,y,z)$, and suppose $\vec{n}$ is a unit vector, normal to the surface at a given point. The component of $\vec{F}(x,y,z)$ in the direction of $\vec{n}$ is given by :



$$\vec{F}_{\vec{n}} =$$

FLUX Integral: $\iint_S \vec{F} \bullet \vec{n}\, dS$

Specific application: Let $\vec{F}(x,y,z) = \rho(x,y,z)\vec{v}(x,y,z)$ where ρ is the fluid density (mass/volume) and $\vec{v}$ is the velocity (length/time). Then the units of $\rho\,\vec{v}$ are ________________________. Multiplying by area gives mass/time or rate of flow.

We will learn a shorter way to compute Flux, but for now all we would need to do is find the unit normals, $\vec{n}$ (with desired orientation), take the dot product with $\vec{F}$ leaving us with a scalar function. So if we call $\vec{F} \bullet \vec{n} = f(x,y,z)$, we are back to 16.7i where we compute surface integrals by computing dS etc.

For surface S given by z=g(x,y) over a region D in the xy plane:

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{{g_x}^2 + {g_y}^2 + 1}\, dA$$

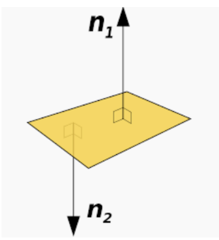
$$\iint_S (\vec{F} \bullet \vec{n})\, dS = \iint_D (\vec{F} \bullet \vec{n})\sqrt{{g_x}^2 + {g_y}^2 + 1}\, dA$$

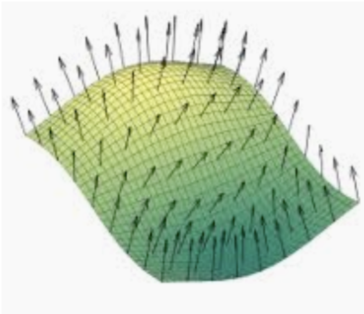
Other orientations similar.

Before we get to an example, let's discuss what we mean by orientation and review how to find unit normals.

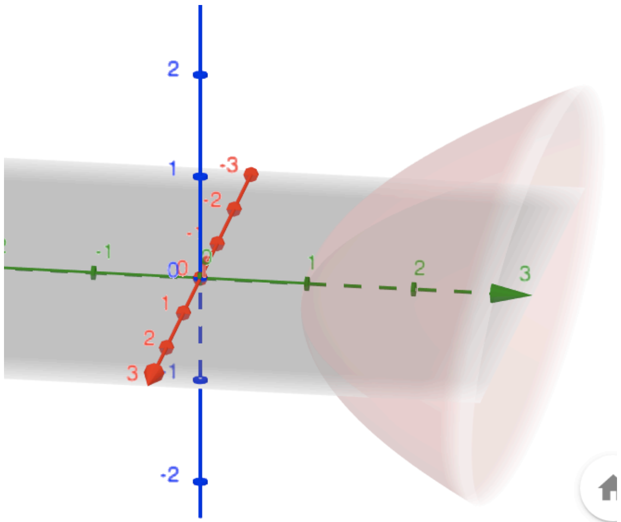
<u>Orientation</u>:
At any point on a surface that has nonzero normal vectors, there are two choices for unit normals. Which one we choose to use in the above formula will determine which direction corresponds to positive Flux.

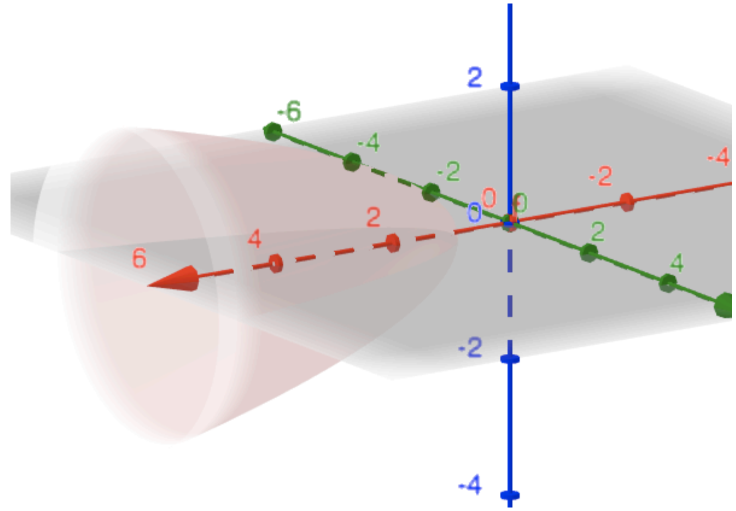
Unless otherwise stated, the convention is to choose the normal that has a positive component for the dependent variable (z in most cases).





Other Orientations:





However, <u>if the surface is closed</u>, the convention for positive orientation is to choose normals that point outward.

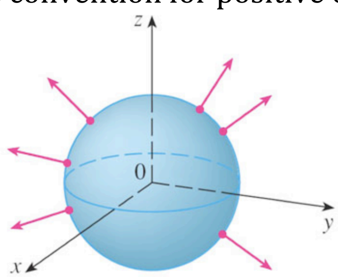


FIGURE 8
Positive orientation

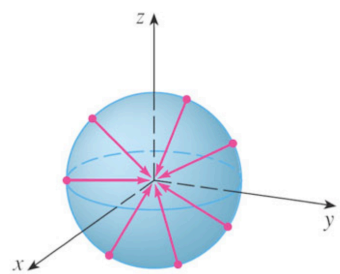


FIGURE 9
Negative orientation

Finding Unit Normals: We found earlier, when finding tangent planes, that if we express a surface as a level surface of a function of 3 variables, then the gradient of that function is normal to the level surface. That is, for the surface G(x,y,z)=k, ∇G is normal to the surface. Then the unit normal $\vec{n} = \frac{1}{\|\nabla G\|}G$.

Example: Find the unit normals for the surface $x^2 + y^2 + z^2 = 1$; z>0

Consider two approaches. One is easier, but the other will lead to a shortcut in computing flux.

1) Let $G(x,y,z) = x^2 + y^2 + z^2$ (so the given surface is a level surface of G) . Then $\vec{n} = \frac{1}{\|\nabla G\|}G = \frac{1}{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}}\langle 2x, 2y, 2z \rangle$

2) This time we will solve for z first to write the surface in function form. $z = \sqrt{1 - x^2 - y^2}$. Then write it in the form $z - \sqrt{1 - x^2 - y^2} = 0$ so we can treat it as a level surface of $G(x,y,z) = z - \sqrt{1 - x^2 - y^2}$. NOW find $\vec{n} = \frac{1}{\|\nabla G\|}\nabla G$.

In general, <u>if we find the unit normals in the second way</u>, writing z in function form as $z = g(x,y) \quad \Rightarrow \quad z - g(x,y) = 0$ and treat this surface as a level surface of the function $G(x,y,z) = z - g(x,y)$ then $\vec{\nabla} G = \langle -g_x, -g_y, 1 \rangle$, thus $\|\vec{\nabla} G\| = \sqrt{(-g_x)^2 + (-g_y)^2 + 1^2}$, so

$$\iint_S (\vec{F} \bullet \vec{n})\, dS = \iint_D \left(\vec{F} \bullet \frac{\vec{\nabla} G}{\|\vec{\nabla} G\|} \right) \sqrt{{g_x}^2 + {g_y}^2 + 1}\, dA = \iint_D \left(\vec{F} \bullet \frac{\qquad\qquad}{\qquad\qquad} \right) \sqrt{{g_x}^2 + {g_y}^2 + 1}\, dA$$

<u>Computing Flux – the shorter way.</u>

For a surface S given by z=g(x,y), IF we write this surface in the form z-g(x,y)=0 and consider it a level surface of the function G(x,y,z)=z-g(x,y), then

$$\iint_S (\vec{F} \bullet \vec{n})\, dS = \iint_D (\vec{F} \bullet \nabla G)\, dA$$

(all in terms of x and y)

will give the flux for an upward orientation. If downward orientation is desired, we use $-\nabla G$ (or just take the opposite of the answer) Note, the z component of ∇G must be 1 for the shortcut to work.

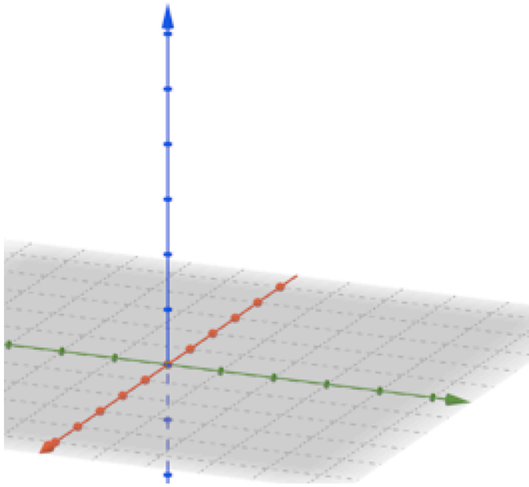
Book notation: If $\vec{F} = \langle P, Q, R \rangle$

$$\text{"}\iint_S \vec{F} \bullet d\vec{S}\text{"} = \iint_S (\vec{F} \bullet \vec{n})\, dS = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

<u>Example</u>: Find the flux for $\vec{F}(x,y,z) = \langle yz, xz, xy \rangle$ given the surface $z = x \sin y; \begin{cases} 0 \le x \le 2 \\ 0 \le y \le \pi \end{cases}$ Positive orientation assumed unless otherwise specified.

Different orientation – Formulas follow logically

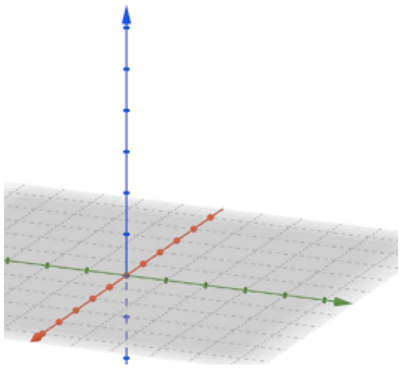
Example: Find $\iint_S \vec{F} \bullet d\vec{S}$ for $\vec{F}(x,y,z) = \langle -x, 2y, -z \rangle$ where S is the portion of $y = 3x^2 + 3z^2$ to the left of y=6 oriented in the positive y direction.



(ans: 24π)

Example: Flux on a closed surface,

Find flux if $\vec{F}(x,y,z)=\langle y,2x,\ z-8\rangle$ where S is the surface of the solid bound by 4x+2y+z=8, z=0, y=0, x=0 with positive orientation.

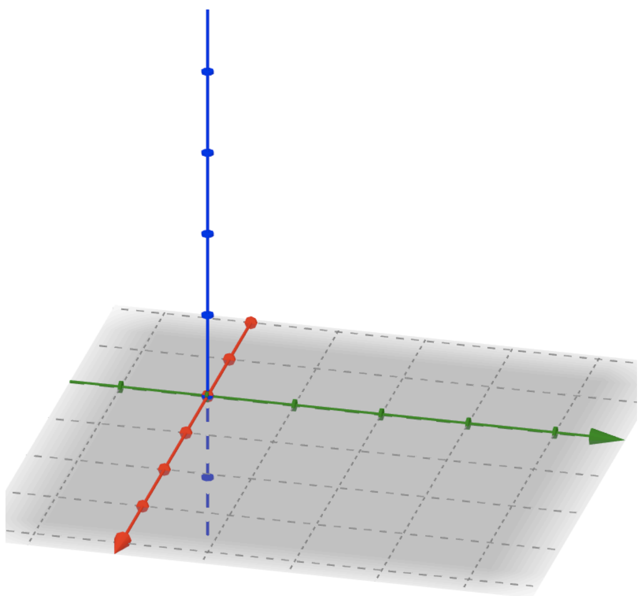


ans:

$$\iint_S \vec{F} \cdot d\vec{S} = \left(\frac{32}{3}\right) + \left(-\frac{32}{3}\right) + \left(-\frac{64}{3}\right) + (32) = \boxed{\frac{32}{3}}$$

16.8 Review of Line Integrals and Stokes' Theorem

Problem: Given $\vec{F}(x,y,z) = \left\langle x^2 4xy^3, y^2 x \right\rangle$, where C is the piecewise smooth curve following the line segments from (0,0,0) to (1,0,0) to (1,3,3) to (0,3,3) and back to (0,0,0), find work.



Review of Line Integrals

Line integral With Respect to Arc Length:

Line integral in a Vector Field – Work:

Directly:

IF $\vec{F}$ is conservative:

1)

2)

If C is a closed curve in ________________________

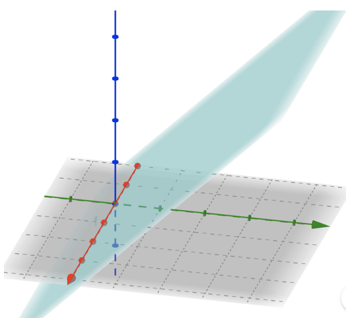
New: If C is a closed curve in ________________________

Stokes' Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

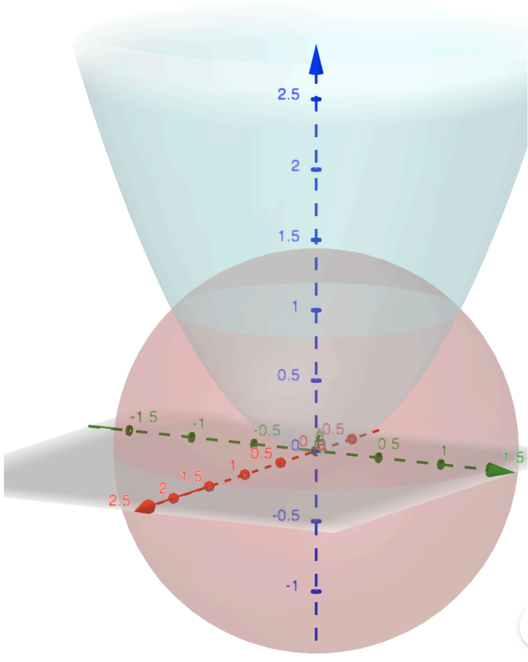
Note: Green's Theorem is a special case
Partial proof in book.

Back to previous problem: Given $\vec{F}(x,y,z) = \left\langle x^2 4xy^3, y^2 x \right\rangle$ where C is the piecewise smooth curve following the line segments from (0,0,0) to (1,0,0) to (1,3,3) to (0,3,3) and back to (0,0,0), find work



One example, many ways:

Given $\vec{F}(x,y,z) = \langle x-y, y-z, z-x \rangle$ where C is the curve of intersection of the sphere $x^2+y^2+z^2=2$ with the paraboloid $z=x^2+y^2$, oriented in the positive direction.



16.9 Review of Flux Integrals and Divergence Theorem

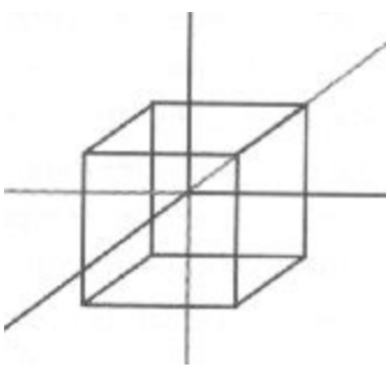
Review of Surface Integrals:

Surface Integral of Scalar Function:

Surface Integral in Vector Field - Flux

HW Problem 16.7: Find Flux

29. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$,
S is the cube with vertices $(\pm 1, \pm 1, \pm 1)$



The Divergence Theorem: A tool for computing flux for a CLOSED surface.

The Divergence Theorem Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

Redo previous problem:

Example from 16.7 notes: Find Flux.

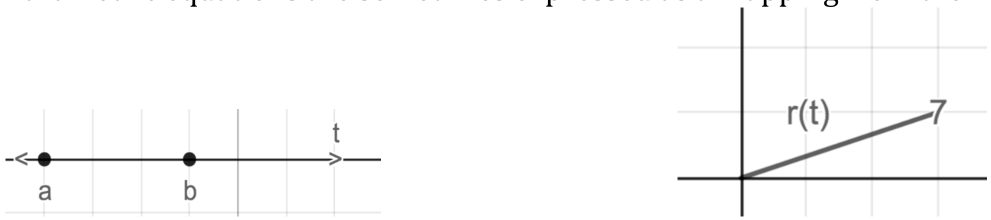
$\vec{F}(x,y,z) = \langle y, 2x,\ z-8 \rangle$ where S is the surface of the solid bound by 4x+2y+z=8, z=0, y=0, x=0 with positive orientation.

16.6 Parametric Surfaces

Parametric Curves:
Recall in R^2, curves can be expressed as an equation in two variables, or as a vector function, equivalent to a pair of parametric equations.

For example: $y=x^2$ can be expressed using the vector function $\vec{r}(t)=\langle t,t^2\rangle$ which is equivalent to C: $\begin{cases} x=t \\ y=t^2 \end{cases}$

Parametric equations are sometimes expressed as a mapping from the number line for t to the xy plane.



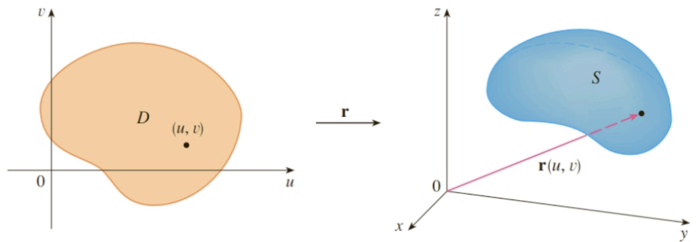
Using parametric equations can be especially helpful in cases where the equation can not be expressed as a function of x or y, but CAN be expressed as functions of t.

Example: $x^2+y^2=4$ could be written as ________________, but parametrically we can write: $\begin{cases} x= \_\_\_\_\_\_\_\_\_\_ \\ y= \_\_\_\_\_\_\_\_\_\_ \end{cases}$

Parametric Surfaces:
Up until now, we have expressed surfaces as equations in 3 variables, which may or may not be expressed in function form. In a way similar to what was done with parametric curves, surfaces can be expressed parametrically as:

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$$

and can be thought as a mapping from R^2 to R^3.



Example: Eliminate the parameters to determine the surface given by: $\vec{r}(u,v) = u\cos v\,\vec{i} + u\,\vec{j} + u\sin v\,\vec{k}$

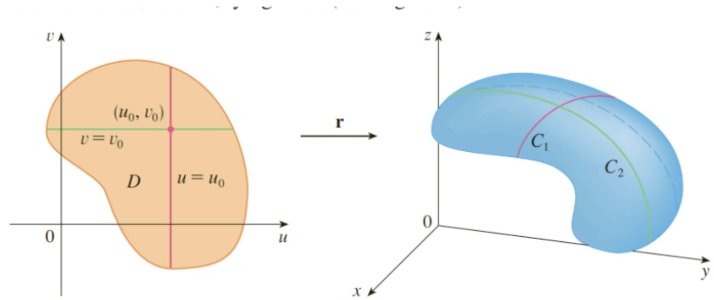
More often, we will be given the equation of the surface, and need to parameterize it.

1) Paraboloid $x = 3y^2 + 2z^2 + z$

2) Cylinder $x^2 + y^2 = 4; \quad 0 \le z \le 1$

3) Sphere: $x^2 + y^2 + z^2 = 25$

Finding Tangent Planes to Surfaces Given Parametrically



If v=v_0, constant, then $\vec{r}(u,v)=\vec{r}(u,v_0)$ is a vector function of one parameter only and so yields a ________________ on the surface, and $\vec{r}\,'(u,v_0)=\dfrac{\partial}{\partial u}\vec{r}(u,v)=\vec{r}_u(u,v)$ which yields ____. Similarly if u=u_0 then $\vec{r}\,'(u_0,v\ \ )=\dfrac{\partial}{\partial v}\vec{r}(u,v)=\vec{r}_v(u,v)$ shown above.

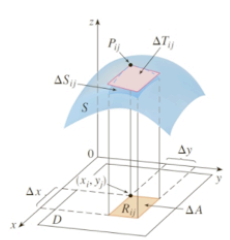
What would we get if we took $\vec{r}_u(u,v)\times\vec{r}_v(u,v)$?

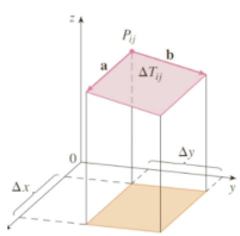
Example: Find the tangent plane to the surface $\vec{r}(u,v)=\langle u,2v^2,u^2+v\rangle$ at (2,2,3)

Surface Integrals with Parametric Surfaces.

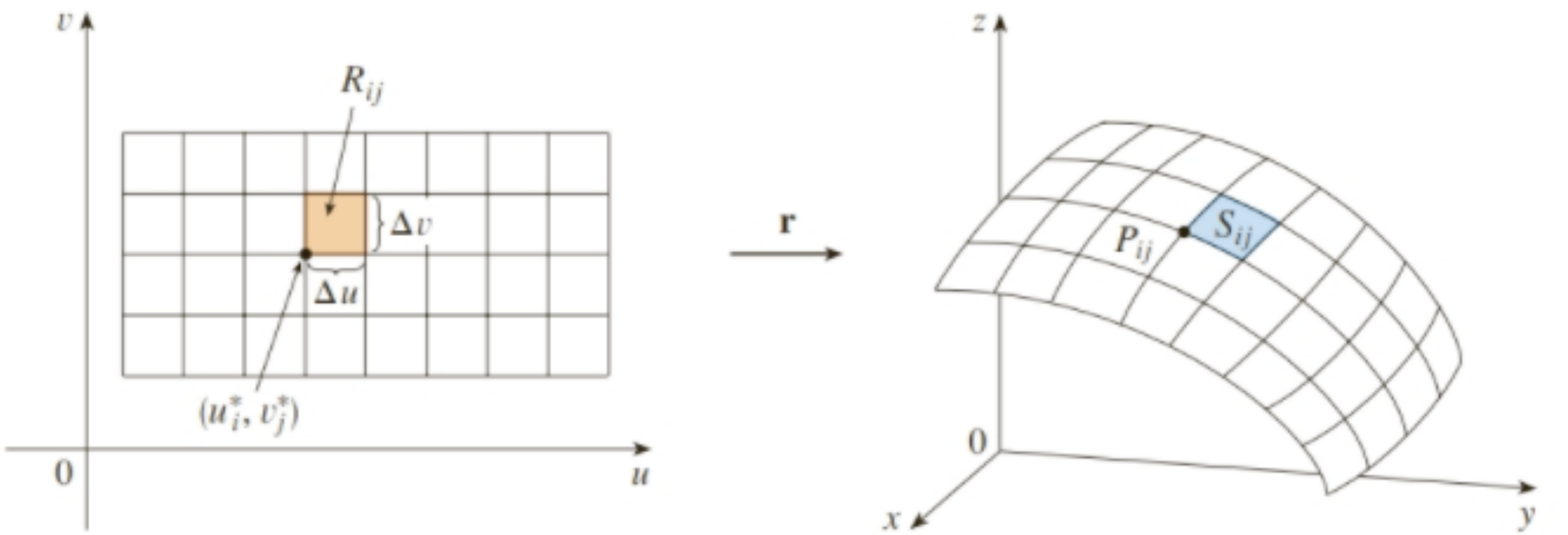
Looking back at our derivation of surface integrals of a scalar function, we needed to determine dS in the integral $\iint_S f(x,y,z)\,dS$

Finding ΔS_{ij}, the area of the ij[th] patch. (See section 15.5)





Using parametric surfaces to find ΔS_{ij}, we already have $\vec{r}_u(u_i^*, v_j^*)$ and $\vec{r}_u(u_i^*, v_j^*)$ tangent to the surface. We just need to find the right length to form the sides of the parallelogram.

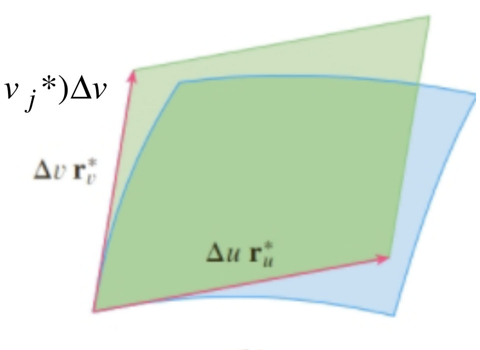


Note that the four corners of R_{ij} map into the four corners of Sij.

By the definition of the partial derivative we find that $\vec{r}_u(u_i^*, v_j^*) \approx \dfrac{\vec{r}(u_i^* + \Delta u, v_j^*) - \vec{r}(u_i^*, v_j^*)}{\Delta u}$ so $\vec{r}_u(u_i^*, v_j^*)\Delta u \approx \vec{r}(u_i^* + \Delta u, v_j^*) - \vec{r}(u_i^*, v_j^*)$ which is the vector we need. Similarly for . $\vec{r}_v(u_i^*, v_j^*)\Delta v$

Then

$$\Delta S_{ij} = \left\| \Delta u\, \vec{r}_u \times \Delta v\, \vec{r}_v \right\| = \left\| \vec{r}_u \times \vec{r}_v \right\| \Delta u \Delta v$$



Thus to do a surface integral in parametric equations, replace dS with ______________________

16.7i EXAMPLES: Surface Integral of a Scalar Function f(x,y,z) over a Surface using parametric surfaces.

For surface S given by z=g(x,y) over a region D in the xy plane, we compute the surface integral by replacing z=g(x,y) into f and using

$$dS = \sqrt{g_x{}^2 + g_y{}^2 + 1}\ dA$$

and dA can be viewed as dydx, dxdy or rdrdθ

$$\iint_S f(x,y,z)dS = \iint f(x,y,g(x,y))\sqrt{g_x{}^2 + g_y{}^2 + 1}\ dA$$

(Recall: The process is similar for x=g(x,y) or y=g(x,z))

If the surface is not easily written in function form but instead is easily represented parametrically, we may prefer the following method.

If the surface S is given parametrically by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ over a domain D in the uv plane, we compute the surface integral by replacing x with x(u,v), y with y(u,v) and z with z(u,v) in f(x,y,z) and using

$$dS = \|\vec{r}_u \times \vec{r}_v\| dA$$

and dA can be viewed as dudv or dvdu.

$$\iint_S f(x,y,z)dS = \iint f(x(u,v),y(u,v),z(u,v)\|\vec{r}_u \times \vec{r}_v\|\ dA$$

<u>Example:</u>

Find the surface area of the portion of the sphere centered at the origin of radius 4, that lies inside the cylinder $x^2 + y^2 = 12$

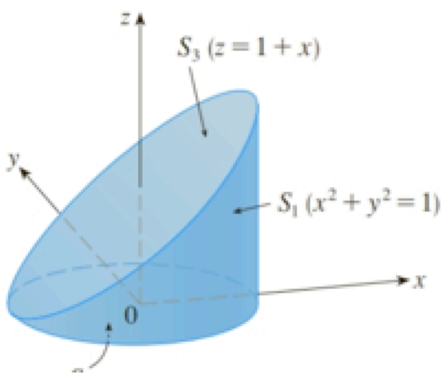
Parameterize sphere:

Find dS= $\|\vec{r}_\theta \times \vec{r}_\phi\| dA$

Revisit Example 3 in the book. We computed this earlier without parametric surfaces. For the top we got $\frac{3\pi}{2}$ and for the

bottom we got 0. Then on the sides we had to break into two surfaces: ________________________ and ________________________ See

Now we are able to use parametric surfaces instead, as done in the book.



EXAMPLE 3 Evaluate $\iint_S z\,dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2 + y^2 = 1$, whose bottom S_2 is the disk $x^2 + y^2 \leqslant 1$ in the plane $z = 0$, and whose top S_3 is the part of the plane $z = 1 + x$ that lies above S_2.

SOLUTION The surface S is shown in Figure 3. (We have changed the usual position of the axes to get a better look at S.) For S_1 we use θ and z as parameters (see Example 16.6.5) and write its parametric equations as

$$x = \cos\theta \qquad y = \sin\theta \qquad z = z$$

16.7ii Examples: FLUX Surface Integral in a VECTOR FIELD over a Surface given by a FUNCTION (using parametrics surfaces)

Recall that without the shortcut, Flux was given by: $\iint_S \vec{F} \bullet d\vec{S} = \iint_S (\vec{F} \bullet \vec{n})\, dS$

If our surface is now given parametrically, we know that $\vec{r}_u \times \vec{r}_v$ is normal to the surface, thus to make a unit normal,

$$\vec{n} = \frac{1}{\|\vec{r}_u \times \vec{r}\|} \vec{r}_u \times \vec{r}_v$$

and the flux integral becomes

$$\iint_S \vec{F} \bullet d\vec{S} = \iint_S (\vec{F} \bullet \vec{n})\, dS = \iint_S (\vec{F} \bullet \frac{1}{\|\vec{r}_u \times \vec{r}\|} \vec{r}_u \times \vec{r}_v)\, \|\vec{r}_u \times \vec{r}\| dA$$

giving us the shorter way to compute Flux:

$$\iint_S \vec{F} \bullet d\vec{S} = \iint_S (\vec{F} \bullet \vec{n})\, dS = \iint_D (\vec{F} \bullet (\vec{r}_u \times \vec{r}_v))dA$$

Example: Compute the flux of $\vec{F}(x,y,z) = \langle x,y,5 \rangle$ across S where S is the boundary of the region enclosed by the cylinder $x^2+z^2=1$ and the planes y=0 and x+y=2 oriented outward

This is a piecewise smooth closed surface, thus we must consider the left side, the right side, and the cylinder sides (top and bottom).

Left side: y=0. As done previously is a function y=g(x,z). Need left pointing (negative y) unit normals.

$$\iint_{left} (\vec{F} \bullet \vec{n})dS = \iint_D (\vec{F} \bullet \nabla G)dA = \iint_D \langle x,0,5 \rangle \bullet \langle 0,-1,0 \rangle dA = 0$$

Right side: y=2-x As done previously is a function y=g(x,z). Need right pointing (positive y) unit normals.

$G = y + x - 2, \quad \vec{\nabla} G = \langle 1,1,0 \rangle$

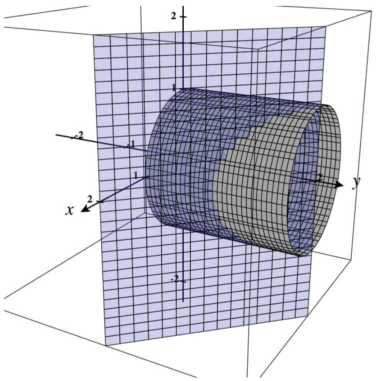
$$\iint_{right} (\vec{F} \bullet \vec{n})dS = \iint_D (\vec{F} \bullet \nabla G)dA = \iint_D \langle x,2-x,5 \rangle \bullet \langle 1,1,0 \rangle dA = \iint_D 2\, dA = 2 areaD = 2\pi$$

Cylindrical Sides

(Can do separately as top and bottom z=g(x,y) or use parametric surfaces and do as one integral

Parameterize S:

$$\begin{cases} x = \cos\theta \\ y = y \\ z = \sin\theta \end{cases} \Rightarrow \vec{r}(y,\theta) = \langle \cos, y, \sin\theta \rangle; \quad 0 \le y \le 2 - x, \quad 0 \le \theta \le 2\pi$$



$\vec{r}_y =$ ____________ $\vec{r}_\theta =$ ________________ $\Rightarrow$

$\vec{r}_y \times \vec{r}_\theta = \langle \cos\theta, 0, \sin\theta \rangle$ (orientation)

$\vec{F}(x,y,z) = \langle x, y, 5 \rangle =$ ____________________

$\vec{F} \bullet (\vec{r}_y \times \vec{r}_\theta) =$ ______________________________

$$\iint_S \vec{F} \bullet d\vec{S} = \iint_S (\vec{F} \bullet \vec{n})\, dS = \iint_D (\vec{F} \bullet (\vec{r}_y \times \vec{r}_\theta))dA = \iint_D (\cos^2\theta + 5\sin\theta)dA$$

Here, $dA = dyd\theta$. We know $0 \le y \le 2 - x, \quad 0 \le \theta \le 2\pi$, and in the parameterization, $x = \cos\theta$ so $0 \le y \le 2 - \cos\theta$

$$\iint_D (\cos^2\theta + 5\sin\theta)dA = \int_0^{2\pi} \int_0^{2-\cos\theta} (\cos^2\theta + 5\sin\theta)dyd\theta = \ldots = 2\pi$$

So the total Flux= $0 + 2\pi + 2\pi = 4\pi$